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ON SOME DISCRETIZATION OF YANG-MILLS EQUATIONS IN MINKOWSKI SPACE

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ABSTRACT. Using methods of differential geometry, a discrete analog of the Yang-Mills equations in Minkowski space is constructed. The gauge transformation law in a discrete formulation is given and gauge invariance of discrete Yang-Mills equations is studied.

1. Introduction.

The main goal of this paper is to construct a gauge-invariant discrete model of Yang-Mills equations in Minkowski space. Based on the formalism described in [1] by Dezin, we consider some intrinsically defined geometric discrete model. A simple two-dimensional discrete model of the classical Yang-Mills equations has been constructed and studied in [2]. However, this discrete model is the lacking of gauge invariance. Some another approaches are proposed in [5, 6]. In [5] a gauge-invariant discrete analog of the Yang-Mills equations is constructed in Euclidean space $\mathbb{R}^n$. We try to define what gauge invariance is in the case of discrete models. The method described in [6] is applicable for obtaining a discrete model of the Yang-Mills equations on the 2-dimensional sphere.

In this paper we concerned with two problems related to the discretization in Minkowski space. First we must determine a combinatorial pseudo-Euclidean space and define a discrete analog of the Lorentz metric. Note that in this case to define discrete analogs of the differential and the exterior multiplication can be used the results of [5]. As in the continual case these operations do not depend on a metric. Secondly, given a discrete analog of the connection 1-form, a discrete covariant derivative must be defined. We always try to be as close to continual Yang-Mills theory as possible. Nevertheless, gauge invariance of the discrete Yang-Mills equations is obtained under some additional conditions (Theorem 1).

It is known that Yang-Mills theory can be regarded as a non-linear generalization of Hodge theory in the 4-dimensional case (see [3]). In Section 5, we construct a formally adjoint operator of the discrete covariant differential. Then we show how to obtain a discrete analog of the generalized Laplace type operator with respect to the Lorentz metric.

2. Preliminaries.

Let $M^4 = \mathbb{R}^{1,3}$ be the Minkowski space-time manifold. Suppose that M^4 has the Lorentz metric $g_{\mu\nu} = diag(-+++)$. Consider the trivial bundle $P = M^4 \times SU(2)$.

Let T^*P be the cotangent bundle of P. It is known (see [4]) that a connection can be shown to arise from a certain 1-form ω belonging to T^*P, where ω is required to have values in the Lie algebra $su(2)$. Let (x, g), $x \in M^4$, $g \in SU(2)$, be local coordinates of the bundle P. Then ω is given by

$$\omega = g^{-1}dg + g^{-1}Ag, \tag{1}$$

where

$$A = \sum_{\alpha,\mu} A_\mu^\alpha(x)\lambda_\alpha dx^\mu. \tag{2}$$

Here we take as a basis for $su(2)$ the set $\{\lambda_\alpha = \frac{\sigma_\alpha}{2i}, \alpha = 1, 2, 3\}$, where σ_α are the standard Pauli matrices. The $su(2)$-valued 1-form A is called the connection form and the functions $A_\mu^\alpha(x)$, connections.

Let the coordinates of P change (locally) from (x, g) to (x', g'). Let us only make a change of fibre coordinates, i.e. $x = x'$ and g' is given by

$$g' = hg, \qquad h \in SU(2). \tag{3}$$

So invariance of ω means that

$$g^{-1}dg + g^{-1}Ag = (g')^{-1}dg' + (g')^{-1}A'g'.$$

Under the change of coordinates (3) the invariant 1-form ω induces a certain transformation law for the connection form A. Taking into account the fact that $dg' = dhg + hdg$ and $dhh^{-1} + hdh^{-1} = 0$, we obtain

$$A' = hdh^{-1} + hAh^{-1}. \tag{4}$$

In Yang-Mills theory this transformation law is called the gauge transformation law.

The curvature 2-form F can be defined as follows

$$F = dA + A \wedge A. \tag{5}$$

We have the tensorial law $F' = hFh^{-1}$ for the change of F under the gauge transformation (4).

Define the covariant exterior differential operator d_A by

$$d_A\Omega = d\Omega + A \wedge \Omega + (-1)^{r+1}\Omega \wedge A, \tag{6}$$

where Ω is a $su(2)$-valued r-form.

Consider the equations

$$d_A F = 0, \tag{7}$$

$$d_A * F = 0, \tag{8}$$

where $*$ is the metric adjoint operation (Hodge star). Equations (7), (8) are called the Yang-Mills equations [3]. Equation (7) is known as the Bianchi identity.

Let Φ, Ψ be $su(2)$-valued r-forms on M^4. The "inner product" can be defined as

$$(\Phi, \Psi) = -tr \int_{M^4} \Phi \wedge *\Psi, \tag{9}$$

where tr is the trace operator. Note that M^4 is non-compact. So all forms referring to the inner product have compact support by assumption.

Then the adjoint operator of d_A can be expressed in the form

$$\delta_A = *^{-1} d_A *,$$

where $*^{-1}$ is the inverse operation to $*$ ($**^{-1} = 1$). Combining the latter with Equation (8) we get

$$\delta_A F = 0.$$

By virtue of (7), this equation is similar to the criterion for a scalar differential form to be harmonic [8]. Thus, if F is a solution of the Yang-Mills equations, then the following Laplace-Beltrami type equation

$$(d_A \delta_A + \delta_A d_A) F = 0 \tag{10}$$

holds immediately on M^4 with respect to the Lorentz metric.

3. Combinatorial model of Minkowski space.

Following [1], let the tensor product $C(4) = C \otimes C \otimes C \otimes C$ of a 1-dimensional complex be a combinatorial model of Euclidean space $\mathbb{R}^4$. The 1-dimensional complex C is defined in the following way. Let C^0 denotes the real linear space of 0-dimensional chains generated by basis elements x_κ (points), $\kappa \in \mathbb{Z}$. It is convenient to introduce the shift operators τ, σ in the set of indices by

$$\tau\kappa = \kappa + 1, \qquad \sigma\kappa = \kappa - 1.$$

We denote the open interval $(x_\kappa, x_{\tau\kappa})$ by e_κ. One can regard the set $\{e_\kappa\}$ as a set of basis elements of the real linear space C^1. Suppose that C^1 is the space of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the introduced spaces $C = C^0 \oplus C^1$. The boundary operator ∂ in C is given by

$$\partial x_\kappa = 0, \qquad \partial e_\kappa = x_{\tau\kappa} - x_\kappa.$$

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements x_κ, e_κ in various way we obtain basis elements of $C(4)$. If c_p, c_q are chains of the indicated dimension, belonging to the complexes being multiplied, then

$$\partial(c_p \otimes c_q) = \partial c_p \otimes c_q + (-1)^p c_p \otimes \partial c_q. \tag{11}$$

Relation (11) defines the boundary operator in $C(4)$.

We suppose that the combinatorial model of Minkowski space has the same structure as $C(4)$. We denote only the basis elements corresponding to the time coordinate of

M^4 by $\bar{x}_\kappa$, $\bar{e}_\kappa$. So, for example, the 1-dimensional basis elements of $C(4)$ can be written as

$$e^1_k = \bar{e}_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes x_{k_4}, \qquad e^2_k = \bar{x}_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes x_{k_4},$$

$$e^3_k = \bar{x}_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes x_{k_4}, \qquad e^4_k = \bar{x}_{k_1} \otimes x_{k_2} \otimes x_{k_3} \otimes e_{k_4}, \tag{12}$$

where $k = (k_1, k_2, k_3, k_4)$ is multiindex, $k_j \in \mathbb{Z}$, $j = 1, 2, 3, 4$.

Let us now consider a dual complex to $C(4)$. We define its as the complex of cochains $K(4)$ with coefficients belonging to $su(2)$ The complex $K(4)$ has a similar structure, namely $K(4) = K \otimes K \otimes K \otimes K$, where K is a dual complex to the 1-dimensional complex C. Basis elements of K can be written as $\{x^\kappa\}, \{e^\kappa\}$. Then an arbitrary basis element of $K(4)$ is given by $s^k = \bar{s}^{k_1} \otimes s^{k_2} \otimes s^{k_3} \otimes s^{k_4}$, where s^{k_j} is either x^{k_j} or e^{k_j}.

As in [2], we define the pairing operation for arbitrary basis elements $\varepsilon_k \in C(4)$, $s^k \in K(4)$ by the rule

$$< \varepsilon_k, as^k >= \begin{cases} 0, & \varepsilon_k \neq s_k \\ a, & \varepsilon_k = s_k, \quad a \in su(2). \end{cases} \tag{13}$$

The operation (13) is linearly extended to cochains. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

The coboundary operator d^c is defined by

$$< \partial\varepsilon_k, as^k >=< \varepsilon_k, ad^c s^k > . \tag{14}$$

The operator d^c is an analog of the exterior differentiation operator.

Let us now introduce in $K(4)$ a multiplication which is an analog of the exterior multiplication for differential forms. First we introduce the r-dimensional complex $K(r)$, $r = 1, 2, 3$, in an obvious notation. Let $s^k_{(p)}$ be an arbitrary p-dimensional basis element of $K(r)$, i.e. the following product $s^k_{(p)} = \bar{s}^{k_1} \otimes ... \otimes s^{k_r}$ contains exactly p of the 1-dimensional basis elements e^{k_j} and $r-p$ of the 0-dimensional basis elements x^{k_j}, $k_j \in \mathbb{Z}$, $j = 1, ...r$. It should be noted that the whole requisite information about the number and situation of "components" is contained in the symbol (p). Then, supposing that the $\cup$-multiplication in $K(r)$ has been defined, we introduce it for basis elements of $K(r+1)$ by the rule

$$(s^k_{(p)} \otimes s^\kappa) \cup (s^k_{(q)} \otimes s^\mu) = Q(\kappa, q)(s^k_{(p)} \cup s^k_{(q)}) \otimes (s^\kappa \cup s^\mu), \tag{15}$$

where $s^k_{(p)}, s^k_{(q)} \in K(r)$, $s^\kappa(s^\mu)$ is either $x^\kappa(x^\mu)$ or $e^\kappa(e^\mu)$, $\kappa, \mu \in \mathbb{Z}$, and the signum function $Q(\kappa, q)$ is equal to -1 if the dimension of both elements s^κ, $s^k_{(q)}$ is odd and to $+1$ otherwise (see [1]). For the basis elements of K the $\cup$-multiplication is defined as follows

$$x^\kappa \cup x^\kappa = x^\kappa, \quad e^\kappa \cup x^{\tau\kappa} = e^\kappa, \quad x^\kappa \cup e^\kappa = e^\kappa, \quad \kappa \in \mathbb{Z},$$

supposing the product to be zero in all other case. To arbitrary forms the $\cup$-multiplication can be extended linearly. Coefficients of forms multiply as matrices.

PROPOSITION 1. *Let φ and ψ be arbitrary forms of $K(4)$. Then*

$$d^c(\varphi \cup \psi) = d^c\varphi \cup \psi + (-1)^p \varphi \cup d^c\psi, \tag{16}$$

where p is the dimension of a form φ.

The proof of Proposition 1 is totally analogous to one in [1, p.147] for the case of discrete forms with real coefficients. □

By definition, the coboundary operator d^c and the $\cup$-multiplication do not depend on a metric. So they have the same structure in $K(4)$ as in the case of the combinatorial Euclidean space [5]. At the same time, to define a discrete analog of the operation $*$ we must take into account the structure of the Lorentz metric on $K(4)$. In this case it is convenient to write the basis elements of the complex $K(4)$ in the form $\bar{\mu}^\kappa \otimes s^k$, where s^k is a basis element of $K(3)$ and $\bar{\mu}^\kappa$ is either $\bar{x}^\kappa$ or $\bar{e}^\kappa$, $\kappa \in \mathbb{Z}$.

Then we define the operation $*$ as follows

$$\bar{\mu}^\kappa \otimes s^k \cup *(\bar{\mu}^\kappa \otimes s^k) = Q(\mu)\bar{e}^\kappa \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}, \tag{17}$$

where $Q(\mu)$ is equal to $+1$ if $\bar{\mu}^\kappa = \bar{x}^\kappa$ and to -1 if $\bar{\mu}^\kappa = \bar{e}^\kappa$.
Relation (17) describes the structure of the Lorentz metric in the discrete model (see [7]).

4. Discrete Yang-Mills equations.

The discrete analog of the connection 1-form (2) can be written as

$$A = \sum_{j=1}^{4} \sum_{k} A_k^j e_j^k, \tag{18}$$

where e_j^k is the 1-dimensional basis element of $K(4)$ and $A_k^j \in su(2)$, $k=(k_1, k_2, k_3, k_4)$, $k_j \in \mathbb{Z}$.

Consider the discrete form

$$h = \sum_{k} h_k x^k, \tag{19}$$

where x^k is the 0-dimensional basis element of $K(4)$ and $h_k \in SU(2)$. Note that the 0-form (19) does not belong to the complex $K(4)$. But, since $x^k \in K(4)$, the $\cup$-multiplication and the coboundary operator d^c are generalized on the forms (19) in an obvious way.

Then the discrete analog of the gauge transformation (3), (4) can be written as

$$g' = h \cup g, \qquad A' = h \cup d^c h^{-1} + h \cup A \cup h^{-1}, \tag{20}$$

where h, h^{-1}, g are forms of the type (19). Here we denote by h^{-1} the form whose coefficients (matrices) are inverse to coefficients of h. If e is the 0-form (19) all of whose coefficients are unit elements of the group $SU(2)$, then we have

$$h \cup h^{-1} = h^{-1} \cup h = e.$$

It should be noted that the 0-forms defined by (19) generate a group by respect to the $\cup$-multiplication.

Given the discrete analog of the 1-form (1) by the formula

$$\omega = g^{-1} \cup d^c g + g^{-1} \cup A \cup g,$$

it is easy to proof that ω is invariant ($\omega = \omega'$) under the transformation (20) (see [5]).

Now consider the 2-form

$$F = \sum_{j=1}^{6} \sum_{k} F_k^j \varepsilon_j^k, \tag{21}$$

where $F_k^j \in su(2)$ and ε_j^k is the 2-dimensional basis element of $K(4)$. The 2-dimensional basis elements of $K(4)$ can be written as follows

$$\varepsilon_1^k = \bar{e}^{k_1} \otimes e^{k_2} \otimes x^{k_3} \otimes x^{k_4}, \qquad \varepsilon_2^k = \bar{e}^{k_1} \otimes x^{k_2} \otimes e^{k_3} \otimes x^{k_4},$$

$$\varepsilon_3^k = \bar{e}^{k_1} \otimes x^{k_2} \otimes x^{k_3} \otimes e^{k_4}, \qquad \varepsilon_4^k = \bar{x}^{k_1} \otimes e^{k_2} \otimes e^{k_3} \otimes x^{k_4},$$

$$\varepsilon_5^k = \bar{x}^{k_1} \otimes e^{k_2} \otimes x^{k_3} \otimes e^{k_4}, \qquad \varepsilon_6^k = \bar{x}^{k_1} \otimes x^{k_2} \otimes e^{k_3} \otimes e^{k_4},$$

where $k_i \in \mathbb{Z}$, $i = 1, 2, 3, 4$.

We define the discrete analog of the curvature 2-form by the formula

$$F = d^c A + A \cup A. \tag{22}$$

PROPOSITION 2. *Under the gauge transformation (20) the curvature form (22) changes as*

$$F' = h \cup F \cup h^{-1}.$$

Proof. The proof closely follows the proof Theorem 2 of [5]. Using (20) and (16) we compute

$$d^c A' = d^c h \cup d^c h^{-1} + d^c h \cup A \cup h^{-1} + h \cup d^c A \cup h^{-1} - h \cup A \cup d^c h^{-1}.$$

Since $d^c e = 0$ by definition of d^c, we have

$$d^c(h \cup h^{-1}) = d^c h \cup h^{-1} + h \cup d^c h^{-1} = 0$$

and so

$$d^c h \cup h^{-1} = -h \cup d^c h^{-1}. \tag{23}$$

Taking into account (23), we obtain

$$A' \cup A' = (h \cup d^c h^{-1} + h \cup A \cup h^{-1}) \cup (h \cup d^c h^{-1} + h \cup A \cup h^{-1})$$
$$= -d^c h \cup d^c h^{-1} + h \cup A \cup d^c h^{-1} - d^c h \cup A \cup h^{-1} + h \cup A \cup A \cup h^{-1}.$$

Then we finally have

$$F' = d^c A' + A' \cup A' = h \cup d^c A \cup h^{-1} + h \cup A \cup A \cup h^{-1}$$
$$= h \cup (d^c A + A \cup A) \cup h^{-1}. \qquad \square$$

From the definition (22) one easily derives that the curvature form F satisfies the identity

$$d^c F + A \cup F - F \cup A = 0. \tag{24}$$

The comparison of (24) with (7) yields a discrete analog of the Bianchi identity. Define now the discrete analog of the exterior covariant differentiation operator by setting

$$d_A^c \Omega = d^c \Omega + A \cup \Omega + (-1)^{r+1} \Omega \cup A,$$

where Ω is an arbitrary r-form of $K(4)$. Then Identity (24) can be rewritten as

$$d_A^c F = 0.$$

In similar manner, we obtain the discrete analog of Equation (8)

$$d_A^c * F \equiv d^c * F + A \cup *F - *F \cup A = 0. \tag{25}$$

Let τ_{ij} (σ_{ij}), $i,j = 1,2,3,4$, $i \neq j$, be the shift operator acting as the operator τ (σ) by the i-th, j-th components of the multiindex $k = (k_1, k_2, k_3, k_4)$. For example,

$$\tau_{12} k = (\tau k_1, \tau k_2, k_3, k_4), \qquad \sigma_{23} k = (k_1, \sigma k_2, \sigma k_3, k_4).$$

Using the definition (17) we compute

$$*F = \sum_k (F^6_{\sigma_{34}k} \varepsilon_1^k - F^5_{\sigma_{24}k} \varepsilon_2^k + F^4_{\sigma_{23}k} \varepsilon_3^k - F^3_{\sigma_{14}k} \varepsilon_4^k + F^2_{\sigma_{13}k} \varepsilon_5^k - F^1_{\sigma_{12}k} \varepsilon_6^k). \tag{26}$$

LEMMA 1. *Let h be a discrete 0-form. Then we have*

$$*(h \cup f) = h \cup *f \tag{27}$$

for an arbitrary p-form $f \in K(4)$.

Proof. Any p-form $f \in K(4)$ can be expressed as

$$f = \sum_k f_k^{(p)} s_{(p)}^k,$$

where $f_k^{(p)} \in su(2)$ and $s_{(p)}^k$ is the p-dimensional basis element of $K(4)$. By definition, we have $x^k \cup s_{(p)}^k = s_{(p)}^k$ for an arbitrary 0-dimensional basis element x^k of $K(4)$. Hence,

$$h \cup f = \left(\sum_k h_k x^k \right) \cup \left(\sum_k f_k^{(p)} s_{(p)}^k \right) = \sum_k h_k f_k^{(p)} s_{(p)}^k.$$

Then we obtain

$$*(h \cup f) = \sum_k h_k f_k^{(p)} * s_{(p)}^k = h \cup \sum_k f_k^{(p)} * s_{(p)}^k = h \cup *f. \quad \square$$

LEMMA 2. *We have*

$$*(f \cup h) = *f \cup h \tag{28}$$

for an arbitrary 2-form $f \in K(4)$ *if and only if coefficients of a 0-form* h *satisfy the following conditions*

$$h_k = h_{\sigma_{ijk}} \tag{29}$$

for all $i, j = 1, 2, 3, 4, i \neq j$.

Proof. From the definition (15) one easily derives that in the form $f \cup h$ two indices in the coefficients h_k are shifted and we have

$$f \cup h = \sum_k (f_k^1 h_{\tau_{12}k} \varepsilon_1^k + f_k^2 h_{\tau_{13}k} \varepsilon_2^k + f_k^3 h_{\tau_{14}k} \varepsilon_3^k + f_k^4 h_{\tau_{23}k} \varepsilon_4^k + f_k^5 h_{\tau_{24}k} \varepsilon_5^k + f_k^6 h_{\tau_{34}k} \varepsilon_6^k).$$

Since

$$*\varepsilon_1^k = -\varepsilon_6^{\tau_{12}k}, \qquad *\varepsilon_2^k = \varepsilon_5^{\tau_{13}k}, \qquad *\varepsilon_3^k = -\varepsilon_4^{\tau_{14}k},$$
$$*\varepsilon_4^k = \varepsilon_3^{\tau_{23}k}, \qquad *\varepsilon_5^k = -\varepsilon_2^{\tau_{24}k}, \qquad *\varepsilon_6^k = \varepsilon_1^{\tau_{34}k},$$

we obtain

$$*(f \cup h) = \sum_k (-f_k^1 h_{\tau_{12}k} \varepsilon_6^{\tau_{12}k} + f_k^2 h_{\tau_{13}k} \varepsilon_5^{\tau_{13}k} - f_k^3 h_{\tau_{14}k} \varepsilon_4^{\tau_{14}k}$$
$$+ f_k^4 h_{\tau_{23}k} \varepsilon_3^{\tau_{23}k} - f_k^5 h_{\tau_{24}k} \varepsilon_2^{\tau_{24}k} + f_k^6 h_{\tau_{34}k} \varepsilon_1^{\tau_{34}k}). \tag{30}$$

Taking into account the relation

$$\sum_k h_k x^k = \sum_k h_{\tau k} x^{\tau k},$$

we compute

$$*f \cup h = \sum_k (-f_k^1 h_{\tau k} \varepsilon_6^{\tau_{12}k} + f_k^2 h_{\tau k} \varepsilon_5^{\tau_{13}k} - f_k^3 h_{\tau k} \varepsilon_4^{\tau_{14}k}$$
$$+ f_k^4 h_{\tau k} \varepsilon_3^{\tau_{23}k} - f_k^5 h_{\tau k} \varepsilon_2^{\tau_{24}k} + f_k^6 h_{\tau k} \varepsilon_1^{\tau_{34}k}), \tag{31}$$

where $\tau k = (\tau k_1, \tau k_2, \tau k_3, \tau k_4)$.

Inserting (30), (31) into (28), we get

$$h_{\tau k} = h_{\tau_{12}k} = h_{\tau_{13}k} = h_{\tau_{14}k} = h_{\tau_{23}k} = h_{\tau_{24}k} = h_{\tau_{34}k}$$

for an arbitrary k. Clearly, these relations imply (29).

On other hand, Conditions (29) we can rewritten as follows $h_{\tau k} = h_{\tau_{ij}k}$ for all $i, j = 1, 2, 3, 4$, $i \neq j$. Substituting the latter into (30) and comparing (30) and (31), we obtain (28). □

It should be noted that in the Lemmas we can taken the 0-form h either as an element of $K(4)$ or as a form of the type (19).

Conditions (29) mean that the "diagonal components" of the 0-form h are equal in all plans. Remind that we regard h as the function over the points x_k and $h|_{x_k} =< x_k, h >= h_k$.

PROPOSITION 3. *The set of 0-forms (19) satisfying Conditions (29) is a group under* $\cup$ *-multiplication.*

Proof. The claim is obvious. By the definition (15), the product of any 0-forms is a 0-form and indices in coefficients do not shift. From this the result follows at once. □

THEOREM 1. *Under Conditions (29) the Yang-Mills equation (25) is gauge invariant.*

Here gauge invariance is understood as follows. If $A(F)$ is a solution of Equation (25), then $A'(F')$ is also a solution of (25).

Proof. By Proposition 3, the form h^{-1} satisfies Conditions (29). Using Proposition 2 from Lemma 1 and 2 we have

$$*F' = h \cup *F \cup h^{-1}.$$

Now express $d^c_{A'} * F'$ in terms of F, A. Applying (16) we compute

$$d^c * F' = d^c h \cup *F \cup h^{-1} + h \cup d^c * F \cup h^{-1} + h \cup *F \cup d^c h^{-1}.$$

Taking into account (20) and (23), we obtain

$$A' \cup *F' = -d^c h \cup *F \cup h^{-1} + h \cup A \cup *F \cup h^{-1}$$

and

$$*F' \cup A' = h \cup *F \cup d^c h^{-1} + h \cup *F \cup A \cup h^{-1}.$$

Thus,

$$d^c_{A'} * F' = h \cup d^c_A * F \cup h^{-1}. \quad \square$$

5. The formally adjoint operator of d^c_A.

Let $V \subset C(4)$ be some fixed "domain" of the complex $C(4)$. We can written V as follows

$$V = \sum_k V_k, \qquad k = (k_1, k_2, k_3, k_4), \quad k_i = 1, 2, ..., N_i, \tag{32}$$

where $V_k = \bar{e}_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes e_{k_4}$ is the 4-dimensional basis element of $C(4)$. We agree that in what follows the subscripts k_i, $i = 1, 2, 3, 4$, always run the set of values indicated in (32). In this section we suppose that coefficients of the discrete forms are vanished on $C(4) \setminus V$. Then the "inner product" for forms $\Phi, \Psi \in K(4)$ of the same degree is defined by the relation

$$(\Phi, \Psi)_V = -tr < V, \Phi \cup *\Psi > . \tag{33}$$

For the forms of different degrees the product (33) is set equal to zero.

The definition imitates correctly the continual case (Relation (9)). It follows from (17) that for the basis elements e^k_i and ε^k_j we have $e^k_1 \cup *e^k_1 = -V^k$, $e^k_i \cup *e^k_i = V^k$ for $i = 2, 3, 4$, $\varepsilon^k_j \cup *\varepsilon^k_j = -V^k$ for $j = 1, 2, 3$ and $\varepsilon^k_j \cup *\varepsilon^k_j = V^k$ for $j = 4, 5, 6$. Then we obtain

$$(A, A)_V = -tr \sum_k \left[-(A^1_k)^2 + (A^2_k)^2 + (A^3_k)^2 + (A^4_k)^2\right]$$

and

$$(F,F)_V = -tr\sum_k \left[-(F_k^1)^2 - (F_k^2)^2 - (F_k^3)^2 + (F_k^4)^2 + (F_k^5)^2 + (F_k^6)^2\right],$$

where A is an 1-form (18) and F is a 2-form (21).

PROPOSITION 4. *Let $\Phi \in K(4)$ be an 1-form and $\Psi \in K(4)$ be a 2-form. Then we have*

$$(d^c\Phi, \Psi)_V = (\Phi, \delta^c\Psi)_V,$$

where

$$\delta^c\Psi = *^{-1}d^c * \Psi \tag{34}$$

is the formally adjoint operator of d^c.

Proof. From (16) and (33) we obtain

$$\begin{aligned}(d^c\Phi,\Psi)_V &= -tr < V, d^c\Phi \cup *\Psi) > \\ &= -tr < V, d^c(\Phi \cup *\Psi) > -tr < V, \Phi \cup d^c * \Psi > \\ &= -tr < \partial V, \Phi \cup *\Psi > -tr < V, \Phi \cup *(*^{-1}d^c * \Psi) > \\ &= -tr < \partial V, \Phi \cup *\Psi > +(\Phi,\ *^{-1}d^c * \Psi)_V,\end{aligned}$$

where we used $**^{-1} = 1$.

Let $\tilde{e}_k^j$, $j = 1,2,3,4$, denote the 3-dimensional basis element of $C(4)$. Using (11) we derive that

$$\begin{aligned}\partial V = \sum_k (&\tilde{e}^1_{\tau N_1,k_2,k_3,k_4} - \tilde{e}^1_{1,k_2,k_3,k_4} - \tilde{e}^2_{k_1,\tau N_2,k_3,k_4} + \tilde{e}^2_{k_1,1,k_3,k_4} \\ &+\tilde{e}^3_{k_1,k_2,\tau N_3,k_4} - \tilde{e}^3_{k_1,k_2,1,k_4} - \tilde{e}^4_{k_1,k_2,k_3,\tau N_4} + \tilde{e}^4_{k_1,k_2,k_3,1}),\end{aligned}$$

where

$$\begin{aligned}\tilde{e}^1_k &= \bar{x}_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes e_{k_4}, \qquad \tilde{e}^2_k = \bar{e}_{k_1} \otimes x_{k_2} \otimes e_{k_3} \otimes e_{k_4}, \\ \tilde{e}^3_k &= \bar{e}_{k_1} \otimes e_{k_2} \otimes x_{k_3} \otimes e_{k_4}, \qquad \tilde{e}^4_k = \bar{e}_{k_1} \otimes e_{k_2} \otimes e_{k_3} \otimes x_{k_4}.\end{aligned}$$

Computing the "boundary components" of the form $\Phi \cup *\Psi$ we obtain the linear combination of the following products:

$$\Phi^j_{k_1\ldots\tau N_i\ldots k_4} \cdot \Psi^r_{k_1\ldots N_i\ldots k_4}, \qquad \Phi^j_{k_1\ldots 0\ldots k_4} \cdot \Psi^r_{k_1\ldots 0\ldots k_4},\ i,j = 1,2,3,4,\ r = 1,2,\ldots,6.$$

Since we have $\Phi^j_{k_1\ldots\tau N_i\ldots k_4} = \Psi^r_{k_1\ldots 0\ldots k_4} = 0$ for all i,j,r by assumption, it follows that $< \partial V, \Phi \cup *\Psi >= 0$. □

For the 2-form F using (26) and the definition of d^c we can rewritten (34) in the form

$$\begin{aligned}\delta^c F = \sum_k [&(\Delta_{k_2}F^1_{\sigma_2 k} + \Delta_{k_3}F^2_{\sigma_3 k} + \Delta_{k_4}F^3_{\sigma_4 k})e^k_1 \\ &+ (\Delta_{k_1}F^1_{\sigma_1 k} + \Delta_{k_3}F^4_{\sigma_3 k} + \Delta_{k_4}F^5_{\sigma_4 k})e^k_2 \\ &+ (\Delta_{k_1}F^2_{\sigma_1 k} - \Delta_{k_2}F^4_{\sigma_2 k} + \Delta_{k_4}F^6_{\sigma_4 k})e^k_3 \\ &+ (\Delta_{k_1}F^3_{\sigma_1 k} - \Delta_{k_2}F^5_{\sigma_2 k} - \Delta_{k_3}F^6_{\sigma_3 k})e^k_4].\end{aligned}$$

Here we denote by $\Delta_{k_i} F^j_k$ the difference $F^j_{\tau_i k} - F^j_k$, $j = 1, 2, .., 6$, and $\tau_i k = k_1..\tau k_i...k_4$, $\sigma_i k = k_1..\sigma k_i..k_4$, $i = 1, 2, 3, 4$.

LEMMA 3. *For any 1-form* $\Phi \in K(4)$ *and 3-form* $\Psi \in K(4)$ *the following relation holds*

$$tr < V, \Phi \cup \Psi >= -tr < V, \Psi \cup * * \Phi > . \tag{35}$$

Proof. The forms Φ and Ψ can be expressed as

$$\Phi = \sum_{i=1}^{4} \sum_{k} \Phi^i_k e^k_i, \qquad \Psi = \sum_{i=1}^{4} \sum_{k} \Psi^i_k \tilde{e}^k_i,$$

where Φ^i_k, $\Psi^i_k \in su(2)$ and $\tilde{e}^k_i$ is the 3-dimensional basis element of $K(4)$.

Using (13), (15) we compute

$$tr < V, \Phi \cup \Psi >= tr \sum_{k} (\Phi^1_{\sigma_1 k} \cdot \Psi^1_k - \Phi^2_{\sigma_2 k} \cdot \Psi^2_k + \Phi^3_{\sigma_3 k} \cdot \Psi^3_k - \Phi^4_{\sigma_4 k} \cdot \Psi^4_k),$$

where $\sigma_i k = (k_1 ... \sigma k_i ... k_4)$.

On the other hand, since

$$* * \Phi = \sum_{i=1}^{4} \sum_{k} \Phi^i_k e^{\tau k}_i, \quad \tau k = (\tau k_1, \tau k_2, \tau k_3, \tau k_4), \tag{36}$$

we have

$$tr < V, \Psi \cup * * \Phi >= tr \sum_{k} (-\Psi^1_k \cdot \Phi^1_{\sigma_1 k} + \Psi^2_k \cdot \Phi^2_{\sigma_2 k} - \Psi^3_k \cdot \Phi^3_{\sigma_3 k} + \Psi^4_k \cdot \Phi^4_{\sigma_4 k})$$

$$= tr \sum_{k} (-\Phi^1_{\sigma_1 k} \cdot \Psi^1_k + \Phi^2_{\sigma_2 k} \cdot \Psi^2_k - \Phi^3_{\sigma_3 k} \cdot \Psi^3_k + \Phi^4_{\sigma_4 k} \cdot \Psi^4_k),$$

where we used $tr(\Phi^i_k \cdot \Psi^i_k) = tr(\Psi^i_k \cdot \Phi^i_k)$. From this the result follows at once. □

It should be noted that in the continual case we have the equality

$$tr(\varphi \wedge \psi) = (-1)^{pq} tr(\psi \wedge \varphi),$$

where φ and ψ are matrix-valued differential forms of degree p, q, respectively. Unfortunately, this equality has not an exact analog in our formalism.

THEOREM 2. *For any 2-form* $F \in K(4)$ *the formally adjoint operator of* d^c_A *acts as follows*

$$\delta^c_A F = *^{-1}(d^c * F - *F \cup * * A + A \cup *F). \tag{37}$$

Proof. We will compute the operator δ^c_A defined by the relation

$$(d^c_A \Phi, F)_V = (\Phi, \delta^c_A F)_V,$$

where Φ is an 1-form.

Using (35) and (33) we have

$$
\begin{aligned}
&(d_A^c\Phi,\ F)_V = -tr < V,\ d_A^c\Phi \cup *F > \\
&= -tr < V,\ d^c\Phi \cup *F > -tr < V,\ (A \cup \Phi \cup *F + \Phi \cup A \cup *F) > \\
&= -tr < V,\ d^c\Phi \cup *F > +tr < V,\ (\Phi \cup *F \cup * * A - \Phi \cup A \cup *F) > \\
&= (d^c\Phi,\ F)_V + tr < V,\ (\Phi \cup * *^{-1}(*F \cup * * A) - \Phi \cup * *^{-1}(A \cup *F)) > \\
&= (\Phi,\ \delta^c F)_V - (\Phi,\ *^{-1}(*F \cup * * A))_V + (\Phi,\ *^{-1}(A \cup *F))_V \\
&= (\Phi,\ \delta^c F - *^{-1}(*F \cup * * A) + *^{-1}(A \cup *F))_V. \quad \square
\end{aligned}
$$

In the continual case, if we choose the Lorentz metric, then

$$* * A = A$$

for an arbitrary differential 1-form A.
Hence the Yang-Mills equation (8) can be rewritten as follows

$$d * F + A \wedge *F - *F \wedge * * A = 0. \tag{38}$$

It follows that a discrete analog of Equation (38) (or (8)) can be given by

$$d_A^c * F \equiv d^c * F + A \cup *F - *F \cup * * A = 0. \tag{39}$$

Comparing the latter and (37) we obtain

$$\delta_A^c F = *^{-1} d_A^c * F.$$

Thus, if the discrete curvature 2-form F is a solution of Equation (39), then the Laplace type equation

$$(d_A^c \delta_A^c + \delta_A^c d_A^c)F = 0$$

holds immediately. This equation we call a discrete analog of Equation (10).

It should be noted that in our discrete model the operation $*^2 = **$ is equivalent to a shift with corresponding sign (see (36)). So, unfortunately, Equation (39) differs from Equation (25). The possibility of involute $(** = 1)$ definition of $*$ is discussed in [6].

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